

# Maximum graphs with unique minimum dominating set of size two

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## Abstract

We prove that the maximum number of edges of a graph of order  $n$  which has a unique minimum dominating set of size two is bounded above by  $\binom{n-2}{2}$ . As a corollary to this result, we prove a conjecture by Fischermann, Rautenbach and Volkmann that the maximum number of edges of a graph which has a unique minimum dominating set of size two is  $\binom{n-2}{2}$ .

## 1 Introduction

Consider a finite, simple graph  $G = (V, E)$ . For any  $v \in V$  let  $N(v, G) = N(v) = \{v' \in V : (v, v') \in E\}$  and for any set  $D \subseteq V$  let  $N(D, G) = N(D) = \cup_{v \in D} N(v, G)$ . Thus  $N(v, G)$  is the collection of vertices which are adjacent or neighbors to  $v$  and  $N(D, G)$  is the collection of all vertices which are adjacent or neighbors to at least one vertex in  $D$ . Similarly for any  $v \in V$  let  $N[v, G] = N[v] = N(v, G) \cup \{v\}$  and for any subset  $D \subseteq V$  let  $N[D, G] = N[D] = \cup_{v \in D} N[v, G]$ . Intuitively  $N(v)$  is the open neighborhood of  $v$  and  $N[v]$  is the closed neighborhood of  $v$ .

A set of vertices  $D \subseteq V$  is a *dominating set* of  $G$  if  $N[D, G] = V$ . A subset of vertices which is a dominating set of minimum cardinality is called a *minimum dominating set* or a  $\gamma$ -set. The *dominating number* of  $G$ , denoted  $\gamma(G)$ , is the cardinality of a minimum dominating set of  $G$ . More precisely, for any finite set  $X$  let  $|X|$  denote the cardinality of  $X$ , thus  $\gamma(G) = \min\{|D| \mid D \text{ is a dominating set of } G\}$ . If the minimum dominating set of  $G$  is unique, then it is called a *unique minimum dominating set* or a unique  $\gamma$ -set. For an overview of results on  $\gamma$ -sets and unique  $\gamma$ -sets see [2], [5], [4], [1], and [3].

We are interested in constructing finite, simple graphs without isolated vertices which have a unique  $\gamma$ -set of cardinality  $\gamma$  using the maximum number of edges. In this paper we prove the following:

**Theorem 1.** *Assume  $n \geq 6$ . Let  $m(n, 2) = m(n)$  be the maximum number of edges of a finite, simple graph  $G$  of order  $n$  without isolated vertices which has a unique  $\gamma$ -set of cardinality 2. Then*

$$m(n, 2) = m(n) \leq \binom{n-2}{2}.$$

Fischermann, Rautenbach and Volkmann [2] generalized the definition of  $m(n, 2)$  to cover dominating sets of cardinality  $\gamma$ . Let  $m(n, \gamma)$  be the maximum number of edges of a finite, simple graph  $G$  of order  $n$  without isolated vertices which has a unique  $\gamma$ -set of cardinality  $\gamma$ . In [2], the authors proved that  $m(n, 1) = \binom{n}{2} - \lceil \frac{n-1}{2} \rceil$ , for  $n \geq 3$ , and made the following conjecture for  $\gamma \geq 2$ .

**Conjecture 2** ([2]). *If  $\gamma \geq 2$  and  $n \geq 3\gamma$ , then  $m(n, \gamma) = \binom{n-\gamma}{2} - \gamma(\gamma - 2)$ .*

In [2], the authors were able to prove the conjecture for the case  $n = 3\gamma$ . Notice there are no graphs of order  $n < 6$  which have a unique  $\gamma$ -set of cardinality 2. Gunther, Hartnell, Markus, and Rall [5] observed that if  $G$  has a unique  $\gamma$ -set  $D$ , then every vertex in  $D$  which is not an isolated point, has at least two *private neighbors*, which are only dominated by that vertex. So if  $v \in D$  is not an isolated point, then there exist at least two points  $a$  and  $b$  such that  $a, b \notin \cup_{v' \in D-v} N[v']$ . Thus a graph with 2 unique guards and no isolated vertices must have  $n \geq 6$ .

One can see that the case where  $G$  has isolated vertices can be handled also. If  $G$  has  $k$  isolated vertices, then remove those  $k$  vertices and then apply the theorem to the subgraph which would consist of  $n - k$  vertices with no isolated points.

Additionally in [2], the authors constructed graphs  $G(n, \gamma)$  which have a unique  $\gamma$ -set of cardinality  $\gamma$  with  $\binom{n-\gamma}{2} - \gamma(\gamma - 2)$  edges. This proves a lower bound for  $m(n, \gamma)$  as follows:

**Theorem 3** (Fischermann, Rautenbach, Volkmann [2]). *If  $\gamma \geq 2$  and  $n \geq 3\gamma$ , then  $m(n, \gamma) \geq \binom{n-\gamma}{2} - \gamma(\gamma - 2)$ .*

Combining Theorem 3 and Theorem 1 we are able to prove the conjecture is true for  $\gamma = 2$ .

**Corollary 4.** *If  $\gamma = 2$  and  $n \geq 6$ , then  $m(n, \gamma) = \binom{n-2}{2}$ .*

## 2 Preliminaries

Throughout this paper we will assume  $G = (V, E)$  is a finite, simple graph with  $n$  vertices, no isolated vertices, and a unique  $\gamma$ -set  $\{x, y\}$ . We will refer to  $x$  and  $y$  as the *unique guards* for  $G$ . Let  $\mathcal{A} = N(x) \setminus N[y]$ ,  $\mathcal{B} = N(y) \setminus N[x]$ , and  $\mathcal{C} = N(x) \cap N(y)$ . For any vertex set  $A \subset V$ , let  $G \setminus A$  denote the subgraph of  $G$  induced by the vertex set  $V \setminus A$ . Let  $G_{xy} = G \setminus \{x, y\}$ .

We show that for any  $G$  as above the number of edges in  $G$  is at most  $\binom{n-2}{2}$ . Or, equivalently, that the total degree of the graph is less than  $(n-2)(n-3) + 2$ .

Denote the *degree* of a vertex  $v \in G$  as  $d(v, G) = d(v)$ . Define the *interior degree* of a vertex  $v \in G_{xy}$  to be  $Id(v) := d(v, G_{xy})$ . Similarly for any set  $D \subseteq V$ , let  $d(D) = \sum_{v \in D} d(v)$  and for  $D \subseteq V(G_{xy})$ , let  $Id(D) = \sum_{v \in D} Id(v)$ .

To prove the result we will make use of the following lemmas:

**Lemma 5.** *Given  $G$  as above, for all  $c \in \mathcal{C}$ , there is a vertex  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  such that  $a, b \notin N(c)$ .*

*Proof.* Assume there is a vertex  $c \in \mathcal{C}$  such that  $\mathcal{A} \subset N(c)$ . Then  $\{c, y\}$  is a  $\gamma$ -set of cardinality two which contradicts our assumption that  $G$  has a unique  $\gamma$ -set of cardinality two. Similarly,  $\mathcal{B} \not\subset N(c)$ .  $\square$

Note that Lemma 5 implies that the interior degree of any element of  $\mathcal{C}$  is at most  $n - 5$ .

**Lemma 6.** *Given  $G$  as above, let  $C = \{c \in \mathcal{C} | Id(c) = n - 5\}$ . Then there exists  $a \in \mathcal{A}$  such that  $a \notin N[C]$  or there exists  $b \in \mathcal{B}$  such that  $b \notin N[C]$ .*

*Proof.* If  $|C| = 1$ , then the proof is complete by Lemma 5. Assume  $|C| \geq 2$ . Choose two vertices  $c, c' \in C$  such that  $a, b \notin N[c]$  where  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$ , and  $a', b' \notin N[c']$  where  $a' \in \mathcal{A}$  and  $b' \in \mathcal{B}$ , and  $b \neq b'$ . If no such pair  $c$  and  $c'$  exist, then there exists a vertex  $b \in \mathcal{B}$  such that  $b \notin N[C]$  which completes the proof. If  $a \neq a'$ , then  $\{c, c'\}$  would be a  $\gamma$ -set which is a contradiction. Therefore we can assume  $a = a'$ . (This completes the case when  $|C| = 2$ .)

If  $|C| > 2$ , it suffices to show that for every other vertex  $v \in C \setminus \{c, c'\}$ ,  $a \notin N[v]$ . Let  $c''$  be any other vertex in  $C \setminus \{c, c'\}$ . Assume  $a'', b'' \notin N[c'']$  where  $a'' \in \mathcal{A}$  and  $b'' \in \mathcal{B}$ . Since  $b' \neq b$ , we can assume without loss of generality that  $b'' \neq b$ . If  $a'' \neq a$ , then  $\{c, c''\}$  would be a  $\gamma$ -set which is a contradiction. Therefore  $a = a''$  which completes the proof.  $\square$

**Lemma 7.** *Given  $G$  as above,  $|\mathcal{A}| \geq 2$  and  $|\mathcal{B}| \geq 2$ .*

*Proof.* If  $\mathcal{A} = \emptyset$ , then either  $\mathcal{C} \neq \emptyset$  or  $(x, y) \in E$  (since  $x$  cannot be isolated). If  $\mathcal{A} = \emptyset$  and there is a  $c \in \mathcal{C}$ , then  $\{c, y\}$  is a  $\gamma$ -set which is a contradiction. If  $\mathcal{A} = \emptyset$  and  $(x, y) \in E$ , then  $\{y\}$  is a  $\gamma$ -set which is a contradiction. Thus  $\mathcal{A} \neq \emptyset$ .

If  $|\mathcal{A}| = 1$  and  $a \in \mathcal{A}$ , then  $\{a, y\}$  is a  $\gamma$ -set which is a contradiction. Thus  $|\mathcal{A}| \geq 2$ . Similarly,  $|\mathcal{B}| \geq 2$ .  $\square$

**Lemma 8.** *Given  $G$  as above, for all  $a \in \mathcal{A}$ , there is an  $a' \in \mathcal{A}$  such that  $a' \notin N[a]$ . For all  $b \in \mathcal{B}$ , there is a  $b' \in \mathcal{B}$  such that  $b' \notin N[b]$ .*

*Proof.* Given  $a \in \mathcal{A}$ , if  $\mathcal{A} \subset N[a]$ , then  $\{a, y\}$  is a  $\gamma$ -set for  $G$ . Similarly, for all  $b \in \mathcal{B}$ ,  $\mathcal{B} \not\subset N[b]$ .  $\square$

Lemma 8 implies that the interior degree of any element of  $\mathcal{A}$  or  $\mathcal{B}$  is at most  $n - 4$ .

### 3 Proving the result

We divide our argument into two main cases based on whether or not  $x$  is adjacent to  $y$ . In section 3.1, Proposition 9 establishes our result for the case where  $(x, y) \notin E$ . Its proof requires cases based on the maximum interior degree of the vertices in  $G_{xy}$ . In section 3.2, Proposition 15 deals with the case where  $(x, y) \in E$ .

#### 3.1 The unique guards are not adjacent

We begin with the following proposition for  $(x, y) \notin E$ .

**Proposition 9.** *Given a graph  $G = (V, E)$  as above, if  $(x, y) \notin E$ , then  $|E| \leq \binom{n-2}{2}$ .*

The proof of Proposition 9 has four cases which are addressed in Lemmas 11 through 14. First we provide sufficient bounds for the sum of the interior degrees of  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $\mathcal{C}$ , then we will use the bounds on the total interior degree to prove the upper bound for  $|E|$ .

**Lemma 10.** *Given a graph  $G = (V, E)$  as above with  $(x, y) \notin E$ , if  $Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) < (n-2)(n-5) + 2 - 2|\mathcal{C}|$ , then  $|E| \leq \binom{n-2}{2}$ .*

*Proof.* Note that the total degree of  $G$ ,  $d(V)$ , is

$$\begin{aligned} d(V) &= d(x) + d(y) + d(\mathcal{A}) + d(\mathcal{B}) + d(\mathcal{C}) \\ &= (|\mathcal{A}| + |\mathcal{C}|) + (|\mathcal{B}| + |\mathcal{C}|) + (Id(\mathcal{A}) + |\mathcal{A}|) + (Id(\mathcal{B}) + |\mathcal{B}|) + (Id(\mathcal{C}) + 2|\mathcal{C}|) \\ &= 2(n-2) + 2|\mathcal{C}| + Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}). \end{aligned}$$

Now the hypothesis leads to the desired bound. □

We now begin to look for the largest interior degree of a vertex in  $G_{xy}$ . As noted above, Lemma 8 implies that the largest interior degree of a vertex in  $\mathcal{A}$  or  $\mathcal{B}$  is  $n-4$ . However, note that there cannot be a vertex  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  such that  $Id(a) = Id(b) = n-4$  since  $a$  and  $b$  would form a  $\gamma$ -set for  $G$ . Also, as a result of Lemma 5, any  $c \in \mathcal{C}$  has  $Id(c) \leq n-5$ . So without loss of generality, we may assume any vertices with interior degree  $n-4$  are elements of  $\mathcal{A}$ .

**Lemma 11.** *Given  $G$  as above with  $(x, y) \notin E$ , if there is at least one vertex in  $G_{xy}$  with interior degree  $n-4$ , then  $|E| \leq \binom{n-2}{2}$ .*

*Proof.* As mentioned above, without loss of generality we may assume that all vertices of interior degree  $n-4$  are in  $\mathcal{A}$ . Let  $A = \{a \in \mathcal{A} | Id(a) = n-4\}$ . By Lemma 8, for every  $a \in A$ , there is exactly one  $a' \in \mathcal{A}$  such that  $a' \notin N[a]$ . Let  $A' = \{a' \in \mathcal{A} | (a, a') \notin E \text{ for some } a \in A\}$ . For any  $a' \in A'$ ,  $N(a') \cap (\mathcal{B} \cup \mathcal{C}) = \emptyset$  since if  $z \in N(a') \cap (\mathcal{B} \cup \mathcal{C})$ , then  $\{a, z\}$  would be a  $\gamma$ -set. So for all  $a' \in A'$ ,  $Id(a') \leq |\mathcal{A}| - 1$ . However for each  $a' \in A'$ ,  $(a', a) \notin E$  for at least one  $a \in A$ . Thus  $Id(A') \leq |A'|(|\mathcal{A}| - 1) - |A|$ . Since all other vertices in  $\mathcal{A}$  have degree  $n-5$  or less, we have the bound

$$Id(\mathcal{A}) \leq |A|(n-4) + [|A'|(|\mathcal{A}| - 1) - |A|] + (|\mathcal{A}| - |A| - |A'|)(n-5). \quad (1)$$

For all  $b \in \mathcal{B}$ , there is some  $b' \in \mathcal{B}$  with  $(b, b') \notin E$ , and for all  $a' \in A'$ ,  $(a', b) \notin E$ . Finally, by Lemma 5, for all  $c \in \mathcal{C}$ , there is a  $b \in \mathcal{B}$  such that  $(c, b) \notin E$  and so we have:

$$Id(\mathcal{B}) \leq |\mathcal{B}|(n-4 - |A'|) - |\mathcal{C}|. \quad (2)$$

Now, by Lemma 5, simply note that every vertex in  $\mathcal{C}$  has interior degree at most  $n-5$ . Thus

$$Id(\mathcal{C}) \leq |\mathcal{C}|(n-5). \quad (3)$$

Combining (1), (2), (3) and  $|\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| = n-2$ , we have

$$\begin{aligned} Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) &\leq |A|(n-4) + [|A'|(|\mathcal{A}| - 1) - |A|] + (|\mathcal{A}| - |A| - |A'|)(n-5) \\ &\quad + |\mathcal{B}|(n-4 - |A'|) - |\mathcal{C}| + |\mathcal{C}|(n-5) \\ &= (n-2)(n-5) + |\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|). \end{aligned}$$

By Lemma 10, it remains to show that  $|\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|) < 2 - 2|\mathcal{C}|$ . If we recall  $n = |\mathcal{A}| + |\mathcal{B}| + |\mathcal{C}| + 2$ , then

$$\begin{aligned} |\mathcal{B}| - |\mathcal{C}| + |A'|(|\mathcal{A}| - n + 4 - |\mathcal{B}|) &= |\mathcal{B}| - |\mathcal{C}| + |A'|(-2|\mathcal{B}| - |\mathcal{C}| + 2) \\ &= |\mathcal{B}|(1 - |A'|) + |A'|(2 - |\mathcal{B}|) - (|A'| + 1)|\mathcal{C}|. \end{aligned}$$

Note by assumption  $|A| \geq 1$  which implies  $|A'| \geq 1$ , so  $|\mathcal{B}|(1 - |A'|) \leq 0$  and  $-(|A'| + 1)|\mathcal{C}| \leq -2|\mathcal{C}|$ . From Lemma 7 we know  $|\mathcal{B}| \geq 2$  so  $|A'|(2 - |\mathcal{B}|) \leq 0$ . Thus by Lemma 10 we have our result.  $\square$

Now we consider the case where there are no vertices of interior degree  $n-4$ . We will consider subcases determined by the highest interior degree present in  $\mathcal{C}$ . First we will consider the case where  $\mathcal{C}$  has at least one vertex of interior degree  $n-5$ .

**Lemma 12.** *Given  $G$  as above with  $(x, y) \notin E$ , if there are no vertices in  $G_{xy}$  with interior degree  $n-4$  and there is at least one vertex in  $\mathcal{C}$  of interior degree  $n-5$ , then  $|E| \leq \binom{n-2}{2}$ .*

*Proof.* Let  $C = \{c \in \mathcal{C} | Id(c) = n-5\}$ . By assumption,  $C \neq \emptyset$ . Fix some  $c \in C$ . By Lemma 5, there exists a vertex  $a \in \mathcal{A}$  such that  $a \notin N(c)$  and a vertex  $b \in \mathcal{B}$  such that  $b \notin N(c)$ . In addition, if  $z \in N[a] \cap N[b]$ , then  $\{z, c\}$  would be a  $\gamma$ -set. Thus  $N[a] \cap N[b] = \emptyset$  which implies that  $Id(a) + Id(b) \leq n-5$ . In fact, by Lemma 6, without loss of generality,  $a \notin N[C]$ . Let  $M = \{v \in C | b \notin N(v)\}$ . Then

$$Id(a) + Id(b) \leq (n-5) - (|M| - 1).$$

By Lemma 5, for all  $v \in C \setminus M$ , there is a  $b' \in \mathcal{B}$  such that  $b' \neq b$  and  $b' \notin N(v)$ . Thus,

$$\begin{aligned} Id(\mathcal{B} \cup \{a\}) &\leq (n-5) - (|M| - 1) + (|\mathcal{B}| - 1)(n-4) - (|C| - |M|) \\ &= (|\mathcal{B}| - 1)(n-4) + (n-5) - (|C| - 1). \end{aligned}$$

Finally, by assumption there are no vertices in  $\mathcal{A}$  of degree  $n-4$  and every vertex in  $C \setminus C$  has degree at most  $n-6$ , so

$$\begin{aligned} Id(\mathcal{A}) + Id(\mathcal{B}) + Id(C) &= Id(\mathcal{A} \setminus \{a\}) + Id(\mathcal{B} \cup \{a\}) + Id(C) + Id(C \setminus C) \\ &\leq (|\mathcal{A}| - 1)(n-5) + (|\mathcal{B}| - 1)(n-4) + (n-5) - (|C| - 1) \\ &\quad + |C|(n-5) + (|C| - |C|)(n-6) \\ &= (n-2)(n-5) - |C| + |\mathcal{B}| - n + 5 \\ &= (n-2)(n-5) - 2|C| + 2 + |C| + |\mathcal{B}| - n + 3 \end{aligned}$$

By Lemma 10, it remains to show that  $|C| + |\mathcal{B}| - n + 3 < 0$ . But recall that  $|\mathcal{A}| + |\mathcal{B}| + |C| = n-2$ , so  $|C| + |\mathcal{B}| - n + 3 = -|\mathcal{A}| + 1$  which is negative by Lemma 7 completing the proof.  $\square$

Now we consider the case where every vertex in  $C$  has interior degree less than  $n-5$  and there is at least one vertex in  $C$  with interior degree  $n-6$ .

**Lemma 13.** *Given  $G$  as above with  $(x, y) \notin E$ , if there are no vertices in  $G_{xy}$  with interior degree  $n-4$ , no vertices in  $C$  of interior degree  $n-5$ , and there is at least one vertex in  $C$  of interior degree  $n-6$ , then  $|E| \leq \binom{n-2}{2}$ .*

*Proof.* Let  $C = \{v \in C \mid Id(v) = n-6\}$ . By assumption,  $C \neq \emptyset$ . Fix some  $c \in C$ , then by Lemma 5 there exist  $a \in \mathcal{A}$ ,  $b \in \mathcal{B}$  and  $z \in G_{xy}$  such that  $a, b, z \notin N[c]$ .

Clearly  $N(a) \cap N(b) \cap N(z) = \emptyset$ . If not, then  $\{w, c\}$  would be a  $\gamma$ -set for any  $w \in N(a) \cap N(b) \cap N(z)$ . Note that among  $a, b$ , and  $z$ , we may connect to every vertex in  $G_{xy}$  at most twice except  $c, a, b$ , and  $z$  (ie. for all  $v' \in G_{xy} \setminus \{a, b, c, z\}$ ,  $|N[v'] \cap \{a, b, z\}| \leq 2$ ). But we may also have up to one edge among  $a, b$ , and  $z$ . Thus

$$Id(a) + Id(b) + Id(z) \leq 2(n-6) + 2 = 2n - 10.$$

We consider several cases based on which set contains  $z$ . First, if  $z \in C \setminus C$ , then there are  $|C| - |C| - 1$  vertices in  $C$  with degree at most  $n-7$  and there are  $|C|$  vertices of degree at most  $n-6$ . Now by assumption, every vertex in  $\mathcal{A} \cup \mathcal{B}$  has degree at most  $n-5$ . This give us the following:

$$\begin{aligned} Id(\mathcal{A}) + Id(\mathcal{B}) + Id(C) &= Id(\mathcal{A} \setminus \{a\}) + Id(\mathcal{B} \setminus \{b\}) + Id(a) + Id(b) + Id(z) \\ &\quad + Id(C) + Id(C \setminus (C \cup \{z\})) \\ &\leq (|\mathcal{A}| - 1)(n-5) + (|\mathcal{B}| - 1)(n-5) + 2n - 10 \\ &\quad + |C|(n-6) + (|C| - |C| - 1)(n-7) \\ &= (n-2)(n-5) - 2|C| + |C| - (n-7). \end{aligned}$$

By Lemma 10, it remains to show that  $|C| - (n - 7) < 2$ . But by Lemma 7,  $|C| \leq n - 6$  so clearly  $|C| < n - 5$  and our result is proved for the case where  $z \in C \setminus C$ .

The cases where  $z$  is an element of  $C$ ,  $\mathcal{A}$ , or  $\mathcal{B}$  are proved with a similar computation.  $\square$

The next lemma considers the case where there are no vertices with interior degree  $n - 4$  and either all vertices in  $C$  have interior degree less than  $n - 6$  or  $C = \emptyset$ . This will complete the case where  $(x, y) \notin E$ .

**Lemma 14.** *Given  $G$  as above with  $(x, y) \notin E$ , if there are no vertices in  $G_{xy}$  with interior degree  $n - 4$ , and either  $C = \emptyset$  or all vertices in  $C$  have interior degree less than  $n - 6$ , then  $|E| \leq \binom{n-2}{2}$ .*

*Proof.* With the given assumptions we have the following bound:

$$\begin{aligned} Id(\mathcal{A}) + Id(\mathcal{B}) + Id(C) &\leq |\mathcal{A}|(n - 5) + |\mathcal{B}|(n - 5) + |C|(n - 7) \\ &= (n - 2)(n - 5) - 2|C|. \end{aligned}$$

So by Lemma 10 we have our result.  $\square$

We see that Lemmas 11, 12, 13, and 14 prove all possible cases when  $(x, y) \notin E$ . Thus we have proved Proposition 9.

### 3.2 The unique guards are adjacent

Here we prove the following proposition for  $(x, y) \in E$ .

**Proposition 15.** *Given a graph  $G = (V, E)$  as above, if  $(x, y) \in E$ , then  $|E| \leq \binom{n-2}{2}$ .*

Before beginning this proof we provide sufficient bounds for the sum of the interior degrees of  $\mathcal{A}$ ,  $\mathcal{B}$ , and  $C$ . The proof of the following lemma is similar to the proof of Lemma 10.

**Lemma 16.** *Given a graph  $G$  as above with  $(x, y) \in E$ , if  $Id(\mathcal{A}) + Id(\mathcal{B}) + Id(C) < (n - 2)(n - 5) - 2|C|$ , then  $|E| \leq \binom{n-2}{2}$ .*

We are now prepared to prove Proposition 15.

*Proof of Proposition 15.* Given  $G$  as above with  $(x, y) \in E$ , note that for any vertex  $a \in \mathcal{A}$ , there exist  $a' \in \mathcal{A}$  and  $b' \in \mathcal{B}$  such that  $a', b' \notin N[a]$  (if not, then either  $\{a, y\}$  or  $\{x, a\}$  would be a  $\gamma$ -set). Thus  $Id(a) \leq n - 5$  for all  $a \in \mathcal{A}$ . Similarly,  $Id(b) \leq n - 5$  for all  $b \in \mathcal{B}$ .

First assume that  $C = \emptyset$ . If all vertices in  $\mathcal{A} \cup \mathcal{B}$  have interior degree  $n - 5$ , then for any vertex  $a \in \mathcal{A}$ , there exists a vertex  $a' \in \mathcal{A}$  and a vertex  $b' \in \mathcal{B}$  such that  $a', b' \notin N[a]$  and  $N[a, G_{xy}] = (\mathcal{A} \cup \mathcal{B}) \setminus \{a', b'\}$ . However  $Id(b') = n - 5$  so there exists a  $b'' \in \mathcal{B}$  such that  $N[b', G_{xy}] = (\mathcal{A} \cup \mathcal{B}) \setminus \{a, b''\}$ . But then  $\{a, b'\}$  would be a  $\gamma$ -set which is a contradiction. This implies that there is a  $v \in \mathcal{A} \cup \mathcal{B}$  such that  $Id(v) \leq n - 6$ . Hence

$$Id(\mathcal{A}) + Id(\mathcal{B}) \leq (n - 3)(n - 5) + (n - 6)$$

and by Lemma 16 we have our result for the case  $|\mathcal{C}| = 0$ .

Assume now that  $\mathcal{C} \neq \emptyset$ . By Lemma 5, for each  $c \in \mathcal{C}$ ,  $Id(c) \leq n - 5$  and there exist  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  such that  $a, b \notin N(c)$ . This restriction on edges between  $\mathcal{C}$  and  $\mathcal{A} \cup \mathcal{B}$  is not accounted for in our original bound on  $Id(\mathcal{A})$  and  $Id(\mathcal{B})$ , so we have the following:

$$Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) \leq (n - 2)(n - 5) - 2|\mathcal{C}|.$$

If there is a  $c \in \mathcal{C}$  with  $Id(c) \leq n - 6$ , then we have

$$Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) \leq (n - 3)(n - 5) + (n - 6) - 2|\mathcal{C}|$$

and by Lemma 16 we have our result for this case.

If all elements of  $\mathcal{C}$  have degree  $n - 5$ , then fix a  $c \in \mathcal{C}$ . There exist  $a \in \mathcal{A}$  and  $b \in \mathcal{B}$  such that  $N[c, G_{xy}] = (\mathcal{A} \cup \mathcal{B} \cup \mathcal{C}) \setminus \{a, b\}$ . As noted above, there must be  $a' \in \mathcal{A}$  and  $b' \in \mathcal{B}$  such that  $a' \notin N[a]$  and  $b' \notin N[b]$ . By assumption,  $Id(c) = n - 5$ , so  $a', b' \in N[c]$ . Since  $\{a', b'\}$  is not a  $\gamma$ -set there exists a vertex  $v \notin N[a'] \cup N[b']$ . The vertex  $v$  cannot be an element of  $\mathcal{C}$  since if it was it would have  $Id(v) = n - 5$  by assumption, and so  $a, b \in N[v]$  making  $\{c, v\}$  a  $\gamma$ -set. If  $v \in \mathcal{A}$  then we have  $a, v \notin N[a']$  and so we see  $Id(a') \leq n - 6$  (since by the above arguments there is also a  $b'' \in \mathcal{B}$  with  $b'' \notin N[a']$ ). Similarly, if  $v \in \mathcal{B}$  then  $Id(b') \leq n - 6$ .

Thus we have an element of  $\mathcal{A} \cup \mathcal{B}$  whose interior degree is bounded by  $n - 6$  before considering the restrictions on edges between  $\mathcal{C}$  and  $\mathcal{A} \cup \mathcal{B}$ , so

$$Id(\mathcal{A}) + Id(\mathcal{B}) + Id(\mathcal{C}) \leq (n - 3)(n - 5) + (n - 6) - 2|\mathcal{C}|$$

and by Lemma 16 we have our result for the final case. □

Propositions 9 and 15 together provide the proof of our main result, Theorem 1. Fischermann, Rautenbach, and Volkmann [2] proved their conjecture for the case  $\gamma = 1$ , and  $n = 3\gamma$ . Corollary 4 proves the conjecture for  $\gamma = 2$ , however, these techniques do not immediately generalize for  $\gamma > 2$ .

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