

FRACTALS IN THE CLASSROOM

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1. ABSTRACT

For the past four years, we have run a workshop for middle and high school teachers introducing Fractal Geometry and its usefulness in the classroom. In this paper, we hope to share our knowledge and passion for the topic. We will introduce the topic of fractals, explain the process of iteration, and give sample exercises.

2. INTRODUCTION

What exactly is a fractal? Traditionally students learn about the familiar forms of symmetry: reflection, translation, rotation. As Frame describes, fractals possess “symmetry under magnification: zooming in on an object leaves the shape approximately unaltered” (Frame et al. 2007). More formally, fractals have the property of *self-similarity*; that is, a fractal is any shape which is made up of smaller copies of itself. Self-similarity is what distinguishes fractals from most conventional Euclidean figures and makes them appealing. Can Euclidean calculations be performed on fractals? Do fractals hold the same characteristics as other Euclidean objects? There is much to explore!

In the past, we have taught students of various levels (age and ability) about Fractal Geometry. Kindergarten and Middle School children were introduced to the idea of self-similarity during different workshops. In addition, we have led in-depth investigations with non-majors in Liberal Arts courses and gone into even more detail with mathematics majors in upper level Geometry courses. The key here is that a teacher has much flexibility with this topic; Fractal Geometry can be tailored to the level of the audience or to time constraints. It can be easily explained, but, at the same time, leads to rich, interesting, and unknown mathematical complexities!

When introducing Geometry, we share with our students the two reasons why we love Geometry: because it is so *old* and because it is so *new*. Euclid’s work has endured over 2000 years. It is solid consistent and timeless. How amazing! However, Geometry is by no means complete. Fractal Geometry is a relatively new area of mathematics, first formalized in the 1960’s – AD, not BC! Many mathematicians today are working in this field. Because fractals are so recent and different, using them to present even old ideas can breathe new life into a classroom. Studying traditional topics, such as measurement, may seem fresh and exciting when applied to fractals. Fractal Geometry challenges students to take a new perspective on their understanding and encourages creativity in their problem solving.

Another motivation for the study of Fractal Geometry comes from nature. Most man-made structures are Euclidean; they have nice straight lines or smooth curves. Nature does not follow the same rules. There are many crevices, wiggles, and jagged edges which give natural objects fractal characteristics. The details of nature can not be truly measured using only Euclidean tools. For instance, if you measured the outline of a tree, a cloud, or a coastline using a Euclidean (straight) ruler, you would obtain different measurements depending on the scale of the ruler. The question

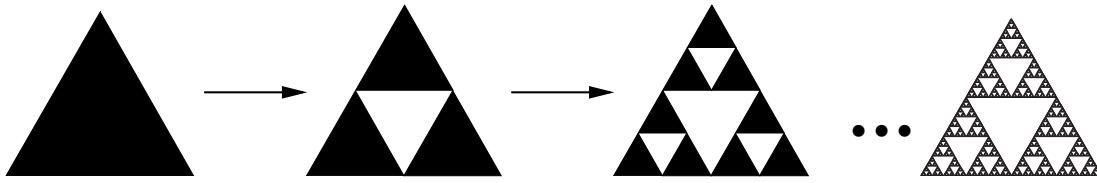


FIGURE 1. Constructing the Sierpinski triangle. At each stage connect the midpoints of each filled in triangle and remove the resulting upside down triangle. Iterate this process.

is not “What did you do wrong?” but rather “Did you use the wrong tool?” Marking off the outline using a stick that is one foot long will give a different result than using a stick that is one inch long. The smaller stick captures more of the detail of the object. In many ways, fractal geometry is a better tool for modeling nature. This is one of the historical motivations for the study of Fractal Geometry.

3. AN APPLICATION

The main tool used to create a fractal is iteration. That is, apply a process, take the result and apply the process again. And again. And again. The result of this is often a mathematical object with striking characteristics. For example, start with a filled-in equilateral triangle. Connect the midpoints of the three sides and remove the small inverted triangle you have just created. You are now left with three smaller filled-in equilateral triangles. Now iterate this process – apply it to each of the smaller triangles. Then repeat with the 9 smaller triangles that result. Then again with the the 27 smaller ones. The limiting shape of this process, the shape that results from infinite repetition, is a fractal known as the Sierpinski triangle. See figure 1 for the construction. Note the self-similarity that results; the shape is made of scaled copies of itself.

Constructing fractals using iteration can appeal to a wide range of students and can be introduced in a short amount of time. You are only required to master a relatively simple process (eg: connecting midpoints and removing a triangle), but when you apply this with a little patience you are rewarded with a very intricate picture.

Once you have constructed a fractal such as the Sierpinski triangle, you can ask several questions about it. For instance, what is its area? You can answer this question in different ways depending on the level of your class. First, assume the area of the original triangle is 1. Then make a table listing the total area of the figure at each stage in the construction. See figure 2.

When comparing the areas at each stage, a pattern emerges. Looking at decimals, it is clear that the areas form a sequence which decreases to zero. Looking at fractions, the pattern of powers becomes evident showing that the area tends to zero. In classes before Calculus, you can simply follow the pattern to its logical conclusion. In Calculus, this example lends itself well to limits. With further thought, most students can find an explanation for why the pattern must hold.

You might also compute the area by asking how much of the original area is removed in the construction process. In the first stage, $\frac{1}{4}$ was removed, in the second $3\left(\frac{1}{4}\right)^2$, and so on. In the n^{th} stage $3^{n-1}\left(\frac{1}{4}\right)^n$ was removed. Now, to find the total area removed simply add all of these fractions. In other words, evaluate the infinite sum $\sum_{n=1}^{\infty} 3^n \left(\frac{1}{4}\right)^n = 1$. This is a great example of a geometric





Stage	Number of Triangles	Area of Each Triangle	Total Area at this Stage	
0	1	1	1	
1	3	$\frac{1}{4}$	$\frac{3}{4} = 0.75$	
2	9	$\frac{1}{16}$	$\frac{9}{16} \approx 0.563$	
3	27	$\frac{1}{64}$	$\frac{27}{64} \approx 0.422$	
n	3^n	$\frac{1}{4^n}$	$\left(\frac{3}{4}\right)^n$	

FIGURE 2. Finding the area of the Sierpinski triangle. Assuming the area of the original triangle is 1, find the area of the shape at each stage in its construction.

series. Since the process started with a triangle of area one and removed exactly the same amount during construction, the Sierpinski triangle is left with zero area.

In either case, the answer seems surprising at first. The Sierpinski triangle is an object with zero area, so does that mean all of its points have been removed and nothing remains? But certainly the edges of the original triangle are never removed. After all, you only removing middle triangles. By the same logic, the edges of the stage 1 triangles must remain in the fractal. In fact, this leads to another question: What is the perimeter of the Sierpinski triangle? Again, this can be answered using a table to look for patterns, logic to explain the patterns, and a variety of tools to answer the question.

4. CONCLUSION

In this article, we have described Fractal Geometry, its motivations, and a sample exercise. However, the Sierpinski triangle is by no means the only fractal which is useful in the classroom. There are many more examples and applications to be found in Fractal Geometry. Some are as easy to construct as the Sierpinski triangle, while others can be as complicated as the Mandelbrot set, which is constructed by iterating a function on the Complex plane. Through Fractal Geometry, students will investigate a range of topics, including number sequences, symmetry, ratio and proportion, measurement, and fractions. At a higher level, tools such as logarithms, composition of functions, Pascal's triangle, arithmetic in different bases, and Complex numbers can be applied.

Fractal activities can be found that address most NCTM standards. Thus, fractals can be taught separately or incorporated as examples into traditional lessons. For additional ideas and detailed lesson plans, please refer to the *Tool Kit of Dynamics Activities* collection published by Key Curriculum Press and the electronic book *Fractal Geometry* by Frame, Mandelbrot, and Neger. We hope that you will consider Fractal Geometry as a resource which reinforces the concepts in your current curriculum and introduces your students to a new and beautiful field of mathematics.

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